Chapter 3: Elementary Functions

The goal of this chapter is to define analytic functions of a complex variable z that reduce to the elementary functions studied in calculus when z is real. Namely,

(1) exponential functions
(2) logarithms
(3) power functions
(4) trig functions + their inverses
(5) hyperbolic trig functions.

We will also develop their basic properties.

Exponential Function

Definition (The Exponential Function) The exponential function e^z or $\exp z$ is defined on $\mathbb{C}$ by the formula

$iy \in \mathbb{C}\setminus\mathbb{R}$ so this is defined by eulers formula.

$$e^z \overset{\text{def}}{=} e^x e^{iy} = e^x \cos y + i e^x \sin y \qquad (z = x+iy).$$

$x \in \mathbb{R}$ so this is the usual exponential function.

Note: if $z = x \in \mathbb{R}$, then $e^z = e^x$ is the usual exponential.

Proposition (Properties of the exponential) Let $z, w \in \mathbb{C}$.

(1) $|e^z| = e^x$ and $\arg e^z = y + 2k\pi$, $k \in \mathbb{Z}$

(2) $e^{z+w} = e^z e^w$

(3) $e^{z-w} = e^z / e^w$

(4) e^z is entire and $\frac{d}{dz} e^z = e^z$

(5) e^z is periodic: $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

Proof.

(1) By definition $e^z = e^x e^{iy}$ is in exponential form so $|e^z| = e^x$ and $\arg e^z = y + 2k\pi$, $k \in \mathbb{Z}$.

(2) Write $z = x+iy$ and $w = u+iv$. Then

$$e^{z+w} = e^{(x+u)+i(y+v)} = e^{x+u} e^{i(y+v)} = e^x e^u e^{iy} e^{iv}$$
$$= e^x e^{iy} e^u e^{iv}$$
$$= e^z e^w.$$

(3) Follows from (2) since

$$e^{z-w} e^w \overset{(2)}{=} e^z.$$

(4) We proved that e^z is entire in an example. $\frac{d}{dz} e^z = e^x$ follows from $f' = u_x + i v_x$.

(5) Follows from (2):

$$e^{z+2k\pi i} \overset{(2)}{=} e^z e^{2k\pi i} = e^z.$$

□

Logarithms

The logarithmic function arises when solving the equation

$$e^w = z \qquad (z \neq 0)$$

for w. Write $z = re^{i\theta}$ and $w = u+iv$. Then

$$e^u e^{iv} = e^w = z = re^{i\theta}.$$

So $e^u = r$ and $v = \underbrace{\theta + 2k\pi}_{\arg z}$, $k \in \mathbb{Z}$.

$u = \ln r = \ln|z|$.

natural log

So $w = \ln|z| + i \arg z$.

Definition (logarithmic function) Following the computation, we define the logarithmic function $\log z$ for any $z \neq 0$ via

$$\log z \overset{\text{def}}{=} \ln|z| + i \arg z.$$

Note: $\log z$ is multiple-valued. The principal branch of $\log z$ is denoted $\operatorname{Log} z$ and is defined by taking the principal argument of z:

$$\operatorname{Log} z \overset{\text{def}}{=} \ln|z| + i \operatorname{Arg} z.$$

The principal branch of log is a single-valued function. //

Proposition (Properties of log / Log)

(1) $e^{\log z} = z$

(2) $\log e^z = z + 2K\pi i$, $K \in \mathbb{Z}$.

(3) $\log z = \operatorname{Log} z + 2K\pi i$, $K \in \mathbb{Z}$

(4) If $z = x$ is a positive real number, then $\operatorname{Log} z = \ln x$.

Proof.

(1)
$$e^{\log z} = e^{\ln|z| + i \arg z} = e^{\ln|z| + i(\operatorname{Arg} z + 2K\pi)}$$
$$= e^{\ln|z|} \cdot e^{i \operatorname{Arg} z} \cdot e^{2K\pi i}$$
$$= |z| e^{i \operatorname{Arg} z} = z.$$

(2)
$$\log e^z = \ln|e^z| + i \arg(e^z)$$
$$= \ln e^x + i(y + 2K\pi) \quad K \in \mathbb{Z}, \quad z = x + iy$$
$$x + iy + 2K\pi i$$
$$= z + 2K\pi i$$

(3)
$$\operatorname{Log} z + 2K\pi i = \ln|z| + i \operatorname{Arg} z + 2K\pi i \qquad (K \in \mathbb{Z})$$
$$= \ln|z| + i(\operatorname{Arg} z + 2K\pi) \qquad (K \in \mathbb{Z})$$
$$= \ln|z| + i(\arg z)$$
$$= \log z.$$

(4) If $z = x > 0$, then
$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$
$$= \ln x.$$

■

Example

(1)
$$\log(1 + \sqrt{3} i) = \ln|1 + \sqrt{3} i| + i \arg(1 + \sqrt{3} i)$$
$$= \ln 4 + i(\pi/3 + 2K\pi), \; K \in \mathbb{Z}.$$

$$\operatorname{Log}(1 + \sqrt{3} i) = \ln 4 + i\pi/3.$$

(2) $\log 1 = \ln|1| + i \arg 1$

$= 0 + i(0 + 2K\pi) = 2K\pi i$, $K \in \mathbb{Z}$

$\text{Log } 1 = 0$ since $\text{Arg } 1 = 0$.

(3) $\log -1 = \ln|-1| + i \arg(-1)$

$= 0 + i(\pi + 2K\pi) = (2K+1)\pi i$

$\text{Log } -1 = \pi i$ since $\text{Arg } -1 = \pi$. //

(4) Familiar properties of logarithms from calculus may not hold:

(a) $\text{Log}\left((-1+i)^2\right) \neq 2\,\text{Log}(-1+i)$

(b) $\log i^2 \neq 2 \log i$

(a) $\text{Log}(-1+i)^2 = \ln|-1+i|^2 + i\,\text{Arg}(-1+i)^2$

$= \ln \sqrt{2}^2 + i(-\pi/2)$

$= \ln 2 - i\pi/2$

$2\,\text{Log}(-1+i) = 2\left(\ln|-1+i| + i\,\text{Arg}(-1+i)\right)$

$= 2\ln\sqrt{2} + 2i\,3\pi/4$

$= \ln 2 + i\,3\pi/2$.

(b) $\log i^2 = \ln|i|^2 + i \arg i^2$

$= 0 + i(\pi + 2K\pi) = i(2K+1)\pi$, $K \in \mathbb{Z}$

$2 \log i = 2(\ln|i| + i \arg i)$

$= 2i(\pi/2 + 2K\pi) = i(4K+1)\pi$, $K \in \mathbb{Z}$ //

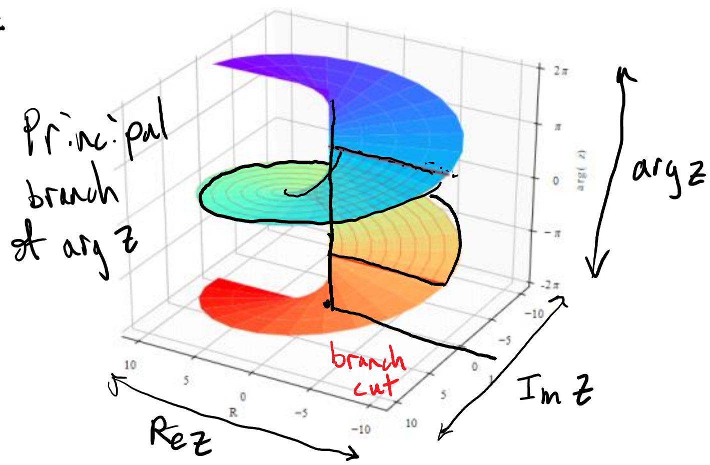
Definition (Branch of a multiple-valued function) A branch of a multiple-valued function f is a single-valued function F that:

(1) is analytic on some domain D;

(2) assigns to each $z \in D$ precisely one value $F(z)$ of $f(z)$.

A portion of a line or curve in the complex plane is called a branch cut for f if a branch of f is defined on it's complement. A point belonging to every branch cut of f is a branch point. //

$f(z) = \arg z$



$\boxed{\text{Proposition (Branches of } \log z)}$ Let $\alpha \in \mathbb{R}$. The function

$$F(z) = \ln r + i\theta, \qquad (r>0,\ \alpha < \theta < \alpha + 2\pi)$$

is a branch of $f(z) = \log z$.

Proof. It is clear that $F(z)$ is single-valued and for each z, $F(z)$ is a value of $\log z$. We need to show that F is analytic. Note that $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ have continuous partial derivatives on the domain of definition.

We have
$$u_r = \frac{1}{r} \qquad v_r = 0$$
$$u_\theta = 0 \qquad v_\theta = 1.$$

Evidently,
$$r u_r = \frac{r}{r} = 1 = v_\theta$$
$$-r v_r = 0 = u_\theta.$$

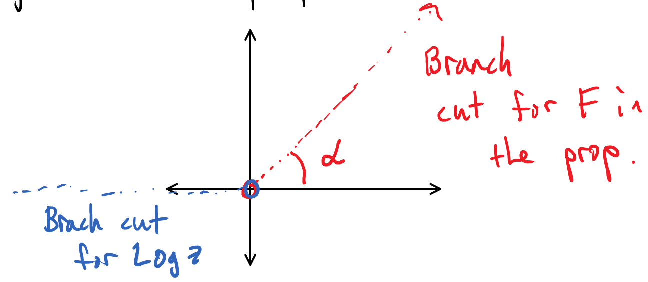
So the Cauchy-Riemann eq are satisfied, hence F is analytic.

In fact
$$\boxed{\begin{aligned}\frac{d}{dz}F(z) &= e^{-i\theta}(u_r + i v_r) \\ &= e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{z}.\end{aligned}}$$

In particular, $\mathrm{Log}\, z$ is a branch of $\log z$ and

$$\boxed{\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}.}$$

The branch cut for $\log z$ in the proposition is the ray $r > 0$, $\theta = \alpha$



The branch cut for $\operatorname{Log} z$ is the ray $r > 0$, $\theta = \pi$. The origin is a branch point for $\log z$. //

Proposition For all $z, w \in \mathbb{C} \setminus \{0\}$,

(1) $\log zw = \log z + \log w$

(2) $\log z/w = \log z - \log w$

These equations are interpreted as follows: given values of two of the logarithms in the equation, there is a value of the third satisfying the eq.

Proof. Compare w/ $\arg zw = \arg z + \arg w$ from Ch 1.

(1) We have
$$\begin{aligned}
\log z + \log w &= \ln|z| + i \arg z + \ln|w| + i \arg w \\
&= \ln|z| + \ln|w| + i\underbrace{(\arg z + \arg w)}_{\arg zw} \\
&= \ln|z||w| + i \arg zw \\
&= \ln|zw| + i \arg zw \\
&= \log zw.
\end{aligned}$$

(2) follows from (1).

The statement does not hold if $\log z$ is replaced w/ $\operatorname{Log} z$.

Example (Integer powers and roots) The logarithmic function can be used to compute integer powers and roots (as previously defined).

(1) $z^n = e^{n \log z}$, $n \in \mathbb{Z}$

(2) $z^{1/n} = e^{1/n \log z}$, $n \in \mathbb{N}$.

For (1),
$$
\begin{aligned}
e^{n \log z} &= e^{n(\ln|z| + i \arg z)} \\
&= e^{n(\ln|z| + i(\operatorname{Arg} z + 2k\pi))} \\
&= e^{n \ln|z|} e^{n i \operatorname{Arg} z} e^{2nk\pi i} \\
&= |z|^n e^{i(n \operatorname{Arg} z)} = |z|^n \left(e^{i \operatorname{Arg} z}\right)^n = \left(|z| e^{i \operatorname{Arg} z}\right)^n \\
&= z^n .
\end{aligned}
$$

polar form of z.

(2)
$$
\begin{aligned}
e^{\frac{1}{n} \log z} &= e^{\frac{1}{n}(\ln|z| + i \arg z)} \\
&= e^{\frac{1}{n}(\ln|z| + i(\operatorname{Arg} z + 2k\pi))} \\
&= \sqrt[n]{|z|}\, e^{i\left(\frac{\operatorname{Arg} z + 2k\pi}{n}\right)} = z^{1/n} .
\end{aligned}
$$

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Power Functions

Definition (Power function) The power function z^c for a fixed complex number $c \in \mathbb{C}$ is the multiple-valued function

$$z^c \overset{\text{def}}{=} e^{c \log z}, \qquad z \neq 0.$$

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Proposition (Branches of z^c) A branch of z^c is determined by specifying a branch of $\log z$:

$$\log z = \ln r + i\theta \qquad (r > 0,\ \alpha < \theta < \alpha + 2\pi).$$

Moreover,

$$\frac{d}{dz} z^c = c\, z^{c-1} \qquad (|z| > 0,\ \alpha < \arg z < \alpha + 2\pi).$$

Proof. We only need to check that z^c is analytic once a branch of $\log z$ has been specified. Since $z^c = e^{c \log z}$ is the composition of two analytic functions e^z and $c \log z$, z^c is analytic by the chain rule. Moreover,

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log z}$$

$$= e^{c\log z} \cdot \frac{d}{dz}(c \log z)$$

$$= \frac{c}{z} e^{c\log z} = c\, \frac{e^{c\log z}}{e^{\log z}} = c\, e^{(c-1)\log z}$$

$$= c z^{c-1}. \quad \blacksquare$$

The principal branch of z^c is defined by specifying the principal branch $\text{Log } z$ of $\log z$. The principal branch of z^c reduces to the usual power function when $z = x \in \mathbb{R}$.

We can define the exponential function with base c by interchanging the roles of z and c.

$\boxed{\text{Definition (exponential function of base c)}}$ The exponential function of base c, $c \neq 0$, is defined via

$$c^z \stackrel{\text{def}}{=} e^{z \log c}$$

Note: c^z is multiple valued since $\log c$ is. When a value of $\log c$ is specified, c^z is entire and

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z\log c} \cdot \frac{d}{dz}(z \log c)$$

$$= c^z \log c.$$

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$\boxed{\text{Question :}}$ what happens if we take $c = e$ (Eulers number)?

Take the principal value $\text{Log } e$ in the definition.

$$e^{z} = e^{z\,\mathrm{Log}\,e} = e^{z(\ln e + i\,\mathrm{Arg}\,e)} = e^{z(1+0)} = e^{z}.$$

//

Example

(1) Compute $i^{i} = e^{i \log i} = e^{i(\ln|i| + i \arg i)}$

$$= e^{i^{2}(\pi/2 + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$= e^{-\pi/2} e^{-2k\pi}, \quad k \in \mathbb{Z}.$$

(2) Compute $(-1)^{1/\pi} = e^{\frac{1}{\pi}\log(-1)}$

$$= e^{\frac{1}{\pi}(\ln|-1| + i \arg -1)}$$

$$= e^{\frac{1}{\pi} i(\pi + 2k\pi)}, \quad k \in \mathbb{Z}$$

$$= e^{i(2k+1)}, \quad k \in \mathbb{Z}.$$

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Trigonometric Functions

Recall, for any $z \in \mathbb{C}$,

$$\mathrm{Re}\, z = \frac{z + \bar{z}}{2} \qquad \mathrm{Im}\, z = \frac{z - \bar{z}}{2i}.$$

Hence, for $x \in \mathbb{R}$,

$$\cos x = \mathrm{Re}(e^{ix}) = \frac{e^{ix} + \overline{e^{ix}}}{2} = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \mathrm{Im}(e^{ix}) = \frac{e^{ix} - \overline{e^{ix}}}{2i} = \frac{e^{ix} - e^{-ix}}{2i}.$$

This suggests a way to extend the domain of definition of the sine and cosine functions to all of $\mathbb{C}$.

Definition (sine and cosine) The sine and cosine functions of a complex variable z are defined via

$$\sin z \overset{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z \overset{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}$$

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By our calculation above, $\sin z$ and $\cos z$ reduce to the ordinary sine and cosine functions when z is real.

Proposition (Analycity of sine and cosine)

(1) $\sin z$ and $\cos z$ are entire

(2) $\frac{d}{dz}\sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$

Proof. (1) $\sin z$ / $\cos z$ are entire since they are linear combinations of entire functions e^{iz}, e^{-iz}.

(2)
$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\frac{d}{dz}\left(e^{iz}-e^{-iz}\right)$$
$$= \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right)$$
$$= \frac{e^{iz}+e^{-iz}}{2} = \cos z.$$

$$\frac{d}{dz}\cos z = \frac{1}{2}\frac{d}{dz}\, e^{iz} + e^{-iz} = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right)$$
$$= i\left(\frac{e^{iz}-e^{-iz}}{2}\right) = -\left(\frac{e^{iz}-e^{-iz}}{2i}\right) \blacksquare$$
$$= -\sin z.$$

Various identies hold. Here are a few:

(1) $\sin -z = -\sin z$

(2) $\cos -z = \cos z$

(3) $\sin z+w = \sin z\cos w + \cos z\sin w$

(4) $\cos z+w = \cos z\cos w - \sin z\sin w$

(5) $\sin 2z = 2\sin z\cos z$

(6) $\cos 2z = \cos^2 z - \sin^2 z$

(7) $\sin^2 z + \cos^2 z = 1$

(8) $\sin(z+2\pi) = \sin z$

(9) $\cos(z+2\pi) = \cos z$

(10) $\sin(z+\pi/2) = \cos z$

(11) $\sin(z-\pi/2) = -\cos z$

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To define the other trig functions, we need to understand the zeros of $\sin z$, $\cos z$. To do this, we need the following new identities:

Proposition

(1) $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$

(2) $\sin z = \sin x \cosh y + i \cos x \sinh y$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

(3) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

Recall:

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

Proof.

(1) $$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= i \sinh y.$$

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y.$$

(2) Write $z = x + iy$. Then

$$\sin z = \sin x + iy \overset{(3)}{=} \sin x \cos iy + \cos x \sin iy \underset{\text{Part (1)}}{=} \underbrace{\sin x \cosh y}_{u} + i\underbrace{\cos x \sinh y}_{v}.$$

Then $$\cos z = \frac{d}{dz} \sin z$$

$$= u_x + i v_x = \cos x \cosh y - i \sin x \sinh y.$$

(3) $$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \cosh^2 y - \sin^2 x \sinh^2 y + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \underbrace{(\cosh^2 y - \sinh^2 y)}_{=1} + \sinh^2 y \underbrace{(\sin^2 x + \cos^2 x)}_{=1}$$

$$= \sin^2 x + \sinh^2 y.$$ ∎

$\boxed{\text{Theorem (Zeros of sine and cosine)}}$ The zeros of $\sin z / \cos z$ are precisely the zeros of the sine and cosine functions of a real variable:

$$\sin z = 0 \quad \text{if and only if} \quad z = k\pi, \ k \in \mathbb{Z}$$

$$\cos z = 0 \quad \text{if and only if} \quad z = k\pi + \pi/2, \ k \in \mathbb{Z}.$$

Proof. Assume $z = k\pi$. Then $\sin z = \sin k\pi = 0$ since $k\pi \in \mathbb{R}$. Similarly, if $z = k\pi + \pi/2$, then $\cos z = \cos k\pi + \pi/2 = 0$.

Conversely, assume $\sin z = 0$. Then

$$0 = |\sin z|^2 = \sin^2 x + \sinh^2 y.$$

Hence, $\sin x = 0$ and $\sinh y = 0$. Hence, $x = k\pi$ and $y = 0$. So $z = k\pi$ as claimed. Now, assume $\cos z = 0$. Then

$$0 = \cos z \overset{(11)}{=} -\sin(z - \pi/2).$$

Hence, $z - \pi/2 = k\pi$. ∎

$\boxed{\text{Definition (tangent, cotangent, secant, cosecant)}}$ The *tangent*, *cotangent*, *secant*, and *cosecant* functions are defined in terms of sine and cosine:

$$\tan z \overset{\text{def}}{=} \frac{\sin z}{\cos z}, \ z \neq k\pi + \pi/2 \qquad \sec z \overset{\text{def}}{=} \frac{1}{\cos z}, \ z \neq k\pi + \pi/2$$

$$\cot z \overset{\text{def}}{=} \frac{\cos z}{\sin z}, \ z \neq k\pi \qquad \csc z \overset{\text{def}}{=} \frac{1}{\sin z}, \ z \neq k\pi$$

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All of these function are analytic on the stated domain since $\sin z, \cos z$ are. Also, they all reduce to the ordinary trig functions when z is real, since sine and cosine do. The derivatives are exactly as expected. //

Hyperbolic Trig Functions

The complex exponential function can be decomposed as a sum of an even and an odd function:

$$e^z = \frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2}$$

We define the hyperbolic cosine and sine functions of a complex variable to be the even and odd part of e^z, respectively:

$$\underline{\cosh} \, z \overset{\text{def}}{=} \frac{e^z + e^{-z}}{2} \qquad \underline{\sinh} \, z \overset{\text{def}}{=} \frac{e^z - e^{-z}}{2}.$$

These functions are entire since e^z and e^{-z} are. and

$$\frac{d}{dz} \sinh z = \cosh z \qquad \frac{d}{dz} \cosh z = \sinh z.$$

They also reduce to the ordinary hyperbolic functions when $z = x \in \mathbb{R}$. //

Proposition (Relation to sine/cosine)

(1) $-i \sinh iz = \sin z$ (3) $\cosh iz = \cos z$

(2) $-i \sin iz = \sinh z$ (4) $\cos iz = \cosh z$.

Proof.

(1)
$$-i \sinh iz = -i \left(\frac{e^{iz} - e^{-iz}}{2} \right) = \frac{e^{iz} - e^{-iz}}{2i} = \sin z.$$

The others are similar. ∎

Corollary (Hyperbolic functions are periodic) The functions $\sinh z$ and $\cosh z$ have a period of $2\pi i$.

Proof. To prove this, we need to show $\sinh z + 2\pi i = \sinh z$.

We have

$$\sinh(z+2\pi i) \overset{(2)}{=} -i\sin(i(z+2\pi i))$$
$$= -i\sin(iz - 2\pi)$$
$$= -i\sin iz$$
$$\overset{(2)}{=} \sinh z.$$

The Proof for cosh is similar. ∎

Proposition (Various Identities)

(1) $\sinh -z = -\sinh z$

(2) $\cosh -z = \cosh z$

(3) $\cosh^2 z - \sin^2 z = 1$

(4) $\sinh(z_1+z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

(5) $\cosh(z_1+z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(6) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

(7) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

(8) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

(9) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

Proof. All can be proved by applying preceding prop. and using ordinary trig identities. To prove (3), start with

$$\sin^2 iz + \cos^2 iz = 1.$$

Then by (2) and (4) of prop.,

$$(-i\sinh z)^2 + \cosh^2 z = 1.$$

Hence, $$\cosh^2 z - \sinh^2 z = 1.$$ ∎

Theorem (zeros of sinh/cosh) The zeros of $\sinh z$ and $\cosh z$ all lie on the imaginary axis. Precisely,

$$\text{(a)} \quad \sinh z = 0 \iff z = k\pi i \ , \ k \in \mathbb{Z}$$
$$\text{(b)} \quad \cosh z = 0 \iff z = (\pi/2 + k\pi) i \ , \ k \in \mathbb{Z}.$$

Proof. (of (a))

$$\begin{aligned}
\sinh z = 0 &\overset{(2)}{\iff} -i \sin iz = 0 \\
&\iff \sin iz = 0 \\
&\iff iz = k\pi \ , \ k \in \mathbb{Z} \\
&\iff z = -k\pi i \ , \ k \in \mathbb{Z} \\
&\iff z = k\pi i \ , \ k \in \mathbb{Z}.
\end{aligned}$$

Proof for $\cosh z$: write $\cosh z$ in terms of $\sinh z$ and apply (a).

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Now that we know the zeros, we can define the hyperbolic tangent:

$$\tanh z \overset{\text{def}}{=} \frac{\sinh z}{\cosh z} \ , \ z \neq (\pi/2 + k\pi) i \ .$$

The rest of the hyperbolic functions are the reciprocals of sinh, cosh, tanh and are defined on the domains specified by the preceding theorem. All are analytic on their domain of definition and the derivatives are as expected. //

Inverse Trig Functions

To find an inverse of $\sin z$, we write $w = \sin^{-1} z$ and try to solve the equation for w:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}.$$

Since $\sin z$ is not one-to-one, the best we can hope for is a multiple-valued function. To solve this equation, multiply through by $2ie^{iw}$:

$$(e^{iw})^2 - 2ize^{iw} - 1 = 0.$$

By quadratic formula (PSet 1 - P1)

$$e^{iw} = \frac{2iz + (-4z^2+4)^{1/2}}{2}$$

$$= iz + (1-z^2)^{1/2}$$

Hence, taking logarithms..

$$\boxed{\sin^{-1} z = w = -i \log(iz + (1-z^2)^{1/2})}$$

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Similar computations for $\cos z$ and $\tan z$ produce:

$$\boxed{\begin{aligned} \cos^{-1} z &= -i \log(z + i(1-z^2)^{1/2}) \\ \tan^{-1} z &= \frac{i}{2} \log \frac{i+z}{i-z} \end{aligned}}$$

A branch of any of these is determined by specifying a branch of the logarithm and a branch of the square root. In that case, the functions are analytic by the chain rule and

$$\boxed{\begin{aligned} \frac{d}{dz} \sin^{-1} z &= \frac{1}{(1-z^2)^{1/2}} \\ \frac{d}{dz} \cos^{-1} z &= -\frac{1}{(1-z^2)^{1/2}} \\ \frac{d}{dz} \tan^{-1} z &= \frac{1}{1+z^2} \end{aligned}}$$

Lets verify the third one:

$$\frac{d}{dz}\tan^{-1} z = \frac{d}{dz}\frac{i}{2}\log\frac{i+z}{i-z}$$
$$= \frac{i}{2}\frac{d}{dz}\log\frac{i+z}{i-z}$$
$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\cdot\frac{d}{dz}\left(\frac{i+z}{i-z}\right)$$
$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\left(\frac{1(i-z)-(-1)(i+z)}{(i-z)^2}\right)$$
$$= \frac{i}{2}\left(\frac{i-z}{i+z}\right)\frac{2i}{(i-z)^2}$$
$$= (-1)\left(\frac{1}{(i+z)(i-z)}\right) = (-1)\left(\frac{1}{-1-z^2}\right) = \frac{1}{1+z^2} \quad //$$

Inverse hyperbolic trig functions can be found in a similar fashion:

$$\boxed{\begin{aligned} \sinh^{-1} z &= \log\left(z + (z^2+1)^{1/2}\right) \\ \cosh^{-1} z &= \log\left(z + (z^2-1)^{1/2}\right) \\ \tanh^{-1} z &= \frac{1}{2}\log\frac{1+z}{1-z}. \end{aligned}}$$

Example As an illustration, we will find all solutions to the equation

$$\sin z = i.$$

The solutions are

$$-i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$z = \sin^{-1} i = -i\log\left(i^2 + (1-i^2)^{1/2}\right)$$
$$= -i\log\left(-1 + 2^{1/2}\right)$$

$$= -i\log\left(-1 \pm \sqrt{2}\right)$$

First look at $\log(-1+\sqrt{2}) = \ln|-1+\sqrt{2}| + i\arg(-1+\sqrt{2})$

$$= \ln(\sqrt{2}-1) + i\,\underbrace{2k}_{\text{even}}\pi, \quad k \in \mathbb{Z}.$$

then look at $\log(-1-\sqrt{2}) = \ln|-1-\sqrt{2}| + i(2l+1)\pi, \quad l \in \mathbb{Z}$

$$= \ln|\sqrt{z}-1| + i\underbrace{2k}_{\text{even}}\pi \quad , k \in \mathbb{Z}.$$

Then look at $\log(-1-\sqrt{z}) = \ln|-1-\sqrt{z}| + i(2l+1)\pi \quad , l \in \mathbb{Z}.$

$$= \ln|1+\sqrt{z}| + i\underbrace{(2l+1)}_{\text{odd}}\pi$$

Then notice that

$$\ln|1+\sqrt{z}| = \ln\left(|1+\sqrt{z}| \frac{|-1+\sqrt{z}|}{|-1+\sqrt{z}|}\right)$$

$$= \ln \frac{1}{|\sqrt{z}-1|} = \ln(|\sqrt{z}-1|)^{-1}$$

$$= -\ln|\sqrt{z}-1|.$$

$$= \boxed{(-1)^{n+1} i \ln(|\sqrt{z}-1|) + \pi n \quad , n \in \mathbb{Z}}$$

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